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Quantum groups and conformal field theories

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Rational conformal field theories can be interpreted as defining quasi-triangular Hopf algebras. The Hopf algebra is determined by the duality properties of the conformal theory.

Important advances have been made recently towards the classification of rational conformal field theories (RCFT). An RCFT is characterized by a chiral algebra $\sigma = \sigma_L \times \sigma_R$ such that σ_L (σ_R) contains at least the identity operator and the Virasoro algebra, and the Hilbert space H of the theory splits into a finite number of irreducible representations of σ: $H = \bigoplus H_i \times H_{\bar{i}}$, with $i, \bar{i}$ running over a finite range of values. Examples are provided by the minimal models of Belavin *et al.* (1988) and the discrete unitary series of Virasoro representations (Friedan *et al.* 1984) whose chiral algebra is the Virasoro algebra; the two-dimensional Wess–Zumino–Witten theory (Witten 1984) with σ_L an affine Kac–Moody algebra, etc. A classification of RCFTs is important in the determination of universality classes of two-dimensional critical systems and it may also be an important step towards the resolution of the far more difficult problem of understanding the space of classical solutions to string theories.

Verlinde (1988) studied the fusion algebra of an RCFT, which is a consequence of the operator algebra of the theory. The structure constants of this algebra are given by the different couplings between three conformal families. If $[\phi_i]$ denotes the conformal family of the primary field ϕ_i, the fusion algebra is written as

$$[\phi_i] \times [\phi_j] = \sum_k N_{ij}{}^k [\phi_k] \tag{1}$$

and the $N_{ij}{}^k$ are non-negative integers. If we define the matrices $(N_i)_j{}^k = N_{ij}{}^k$, the associativity of the operator algebra of the conformal theory implies that the N_is commute. More abstractly, the fusion algebra is a commutative associative algebra with as many generators as conformal families in the theory and with structure constants $N_{ij}{}^k$. For each family $[\phi_i]$ we can construct its character:

$$\chi_i(\tau) = \mathrm{Tr}_{[\phi_i]}\, q^{L_0 - \frac{1}{24}c}, \quad q = \mathrm{e}^{2\pi \mathrm{i}\tau}. \tag{2}$$

The behaviour of (2) under modular transformations in a modular covariant theory is:

$$\left.\begin{aligned} T:&\quad \chi_i(\tau+1) = \mathrm{e}^{2\pi\mathrm{i}(h_i - \frac{1}{24}c)}\chi_i(\tau), \\ S:&\quad \chi_i(-1/\tau) = S_i^{\,j}\chi_j(\tau), \end{aligned}\right\} \tag{3}$$

where h_i is the conformal dimension of ϕ_i, and c is the central extension of the Virasoro algebra. Verlinde (1988) showed by many examples that the matrix S diagonalizes the fusion rules. More precisely, if we write $N_{ij}{}^0 = C_{ij}$ and use C to lower indices, then

$$N_{ijk} = \sum_m \frac{S_{im} S_{jm} S_{km}}{S_{0m}} \tag{4}$$

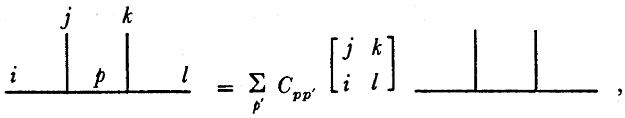
and the eigenvalues of N_i are $\lambda_i^{(k)} = S_i^{\,k}/S_0^{\,k}$. This striking connection between modular transformations and the fusion rules was proved rigorously by Moore & Seiberg (1989) using a set of polynomial equations characterizing RCFT. The polynomial equations involve the matrices S and T and two other matrices C and N expressing the duality properties of the tree-level conformal blocks. To exhibit the equations satisfied by C and N it is convenient to introduce chiral vertices (see, for example, Schroer 1987, and references therein; Fröhlich 1987; Tsuchiya & Kanie 1987). They are operators which represent the holomorphic three-point functions:

$$\Phi\{^{\,i}_{jk}\}(z):\quad H_k \rightarrow H_j, \tag{5}$$

where i is an index for a primary field. The conformal blocks can be written in terms of expectation values of products of chiral vertices. For example:

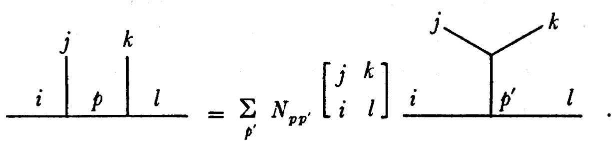
$$\mathscr{F}^{ijkl}_{p}(z,w) = \langle i|\, \Phi\{^{\,j}_{ip}\}(z)\, \Phi\{^{\,k}_{pl}\}(w)\, |l\rangle. \tag{6}$$

The matrix C describes the exchange of two chiral vertices. At the level of vertices it is the matrix representing the braiding (through analytic continuation) of the j,k legs of (6). Graphically,



(7)

where the left-hand side is a pictorial representation of the block, $\mathscr{F}_p^{\,ijkl}$. The matrix N is a consequence of the associativity of the operator product expansion:



(8)

The hexagon equation for C follows from the defining relations of the braid group, and the pentagon equation satisfied by N is a consequence of the associativity of the operator product expansion (see Moore & Seiberg (1989) for details). The proof of the Verlinde conjecture is a consequence of the pentagon identity after one writes a precise representation of the Verlinde operators. These are defined as follows. On the torus we choose a homology basis (a,b). For an RCFT given a primary field ϕ_i, we can always find a conjugate field $\phi_{\bar{\imath}}$ such that the operator product $\phi_i \times \phi_{\bar{\imath}}$ contains the identity. The Verlinde operators $\phi_i(a), \phi_j(b)$ act on the characters $\chi_i(\tau)$. They correspond to inserting the identity factorized into $\phi_i, \phi_{\bar{\imath}}$, then taking ϕ_i around the a- or b-cycle, and finally recombining the two operators into the identity once again. If the a-cycle represents the equal-time surface, the action of $\phi_i(a)$ on χ_j is diagonal:

$$\phi_i(a)\,\chi_j(\tau) = \lambda_i^{(j)}\chi_j(\tau). \tag{9a}$$

The action of $\phi_j(b)$ is more complicated;

$$\phi_i(a)\,\chi_j(\tau) = A_{ij}^{\;\;k}\chi_k(\tau), \tag{9b}$$

because a and b are exchanged by the modular transformation S, then $\phi_i(b) = S\phi_i(a)\,S^{-1}$, and the matrices $(A_i)_j^{\,k} \equiv A_{ij}^{\;\;k}$ all commute. In fact, S diagonalizes the A_is. The conjecture by Verlinde was that A_i and N_i coincide.

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There are two ways that quantum groups (Drinf'eld 1986) enter into conformal field theory. First, the Verlinde operators are associated to closed paths on the torus. It is possible to construct analogues to these operators for open paths at tree level (Alvarez-Gaumé *et al.* 1989*a*). The advantage is that these operators are compatible with the operation of taking traces, and they also imply the hexagon and pentagon equations, hence the proof of the Verlinde conjecture becomes conceptually simpler. Furthermore, these open Verlinde operators satisfy the defining relations for a quantum group. The second application of quantum groups has to do with the fact that they provide solutions to the polynomial equations (Alvarez-Gaumé *et al.* 1988). Moreover, all the known RCFTS can be obtained by using the Goddard–Kent–Olive (GKO) construction (Goddard *et al.* 1986) for appropriate groups G, H, $H \subset G$, and their C, N matrices and modular properties seem to follow from the quantum deformations of G and H (Alvarez-Gaumé *et al.* 1989*b*). Whether the complete set of solutions to the polynomial equations are given by quantum groups is not clear at present.

In the first application we begin with the space of chiral vertices V compatible with the fusion rules $N_{ij}{}^{k}$. On V we can define two operations: one is sewing, denoted by *, and the other one is taking characters. We want to find an algebra of automorphisms of V compatible with (i) sewing, (ii) braiding, (iii) the N operation (8) or s–t duality. Denoting collectively by s the three labels in the chiral vertex (5), the braiding (or exchange) operation can be written as

$$\Phi_{S_1}(z)\,\Phi_{S_2}(w) = R_{S_1 S_2 S_3 S_4}\,\Phi_{S_3}(w)\,\Phi_{S_4}(z). \qquad (10)$$

If Q is the automorphism algebra and $X \in Q$, we can write the action on V as

$$X(\Phi_{S_1}) = \sum_{S_2} C^{(1)}_{S_1 S_2}\,\Phi_{S_2}. \qquad (11)$$

If we think of R as acting on $V \otimes V$, condition (i) becomes

$$RC_1 C_2 = C_2 C_1 R, \quad C_1 = C \otimes 1, \quad C_2 = 1 \otimes C, \qquad (12)$$

reminiscent of the definition of a quantum group (Drinf'eld 1986). If all we had was (12), we could always define a representation of Q by using the adjoint representation (in analogy with ordinary Lie algebras). The problem is that the constraint imposed by the fusion rules,

$$\Phi\left\{{}^{ij}_{k}\right\}\Phi\left\{{}^{l}_{mn}\right\},$$

vanishes unless $k = l$. Nevertheless, for every primary field we can construct an element of Q. Choosing for simplicity a self-conjugate primary field (i.e. $\phi_a \times \phi_a = 1 + \ldots$) we obtain

$$X^{a}\left(\Phi\left\{{}^{j}_{ik}\right\}\right) = \sum_{m,n} R\binom{a}{mi}\binom{j}{ik}\binom{j}{mn}\binom{a}{nk}\,\Phi\left\{{}^{j}_{mn}\right\}. \qquad (13)$$

Graphically, we can represent the chiral vertex as in figure 1, with the representations H_i and H_k attached in the open boundaries and the primary field ϕ_j at the point z. The operator (13) can be interpreted as in figure 2; we factorize the identity into ϕ_a and $\phi_{\bar{a}}$ and then move the fields to the i, k ends along the path shown. For more complicated surfaces one can do the same operation for each open path. This is why we can interpret X^a as the open Verlinde operators. The conditions (i), (ii) and (iii) imply also the tree-level polynomial equations (see Alvarez-Gaumé *et al.* 1989*a* for details). Finally, the action on the characters is based on the operation of taking traces. In particular, the characters are, schematically,

$$\chi_i = \mathrm{Tr}_{H_i}\, q^{L_0} \underset{i\ \ i}{\perp}{}^{id}, \qquad (14)$$

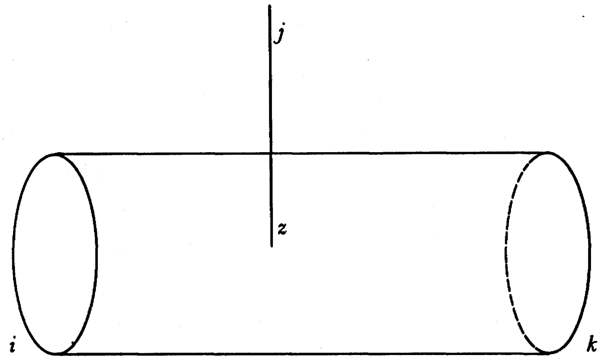

FIGURE 1. Graphical representation of the chiral vertex.

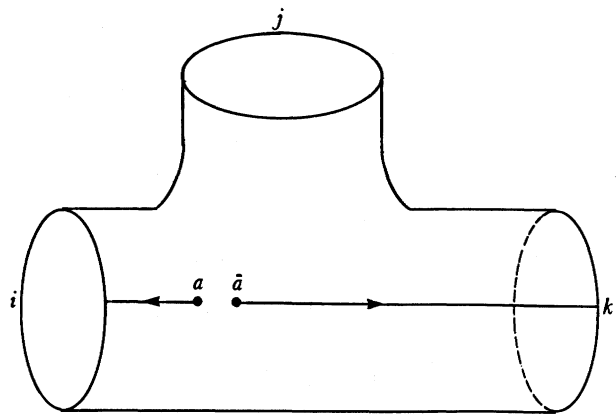

FIGURE 2. Interpretation of the operator defined by equation (13).

where we have represented the chiral vertex by $i \perp^{j} k$. The action of X^a on χ_i is defined by first acting on the chiral vertex and then taking the trace. It is easy to see, by using (13) and the triviality of braiding with the identity, that

$$X^a(\chi_j) = \sum_m N_{aj}{}^k \chi_k, \tag{15}$$

which is indeed the standard Verlinde operator.

The second use of quantum groups is related to the solutions of the polynomial equations. This can be illustrated with the quantum group SL$(2,q)$ which is associated to the level k Wess–Zumino–Witten theory (WZW) (see Alvarez-Gaumé *et al.* 1988). This algebra satisfies the defining relations:

$$[X^+, X^-] = (q^{\frac{1}{2}H} - q^{-\frac{1}{2}H})/(q^{\frac{1}{2}} - q^{-\frac{1}{2}}), \quad [H, X^{\pm}] = \pm 2X^{\pm}. \tag{16}$$

When q is an arbitrary real or complex number the representation theory of this algebra is analogous to the classical case. In RCFT, q is a root of unity. For instance, in the WZW theory, $q = \exp(2\pi i/k+2)$. Now the representation theory becomes more interesting. The only regular representations have spin $j = 0, \frac{1}{2}, 1, \ldots, \frac{1}{2}k$, as in the WZW theory. Furthermore, the composition of angular momentum generates the WZW fusion rules,

$$[j_1] \times [j_2] = \sum_{j=|j_1-j_2|}^{\min(j_1+j_2,\, K-j_1-j_2)} [j], \tag{17}$$

and the C and N matrices are given by the q-analogues of the $6-j$ symbols computed in Kirillov & Reshestikhin (1988),

$$C_{jj'}\begin{bmatrix} j_2 & j_3 \\ j_1 & j_4 \end{bmatrix} = (-1)^{j+j'-j_1-j_4} q^{\frac{1}{2}(C_{j_1}+C_{j_4}-C_j-C_{j'})} \begin{Bmatrix} j_2 & j_1 & j \\ j_3 & j_4 & j' \end{Bmatrix}_q,$$

$$N_{jj'}\begin{bmatrix} j_2 & j_3 \\ j_1 & j_4 \end{bmatrix} = \begin{Bmatrix} j_1 & j_2 & j \\ j_3 & j_4 & j' \end{Bmatrix}_q.$$

In this way, one reproduces the results of Schroer (1987), Fröhlich (1987) and Tsuchiya & Kanie (1987), and, by using the q-characters of $SL(2,q)$, one finds that the modular transformations are represented by $q \rightarrow q^{-1}$.

A plausible reason why the quantum group appears is because the polynomial equations are concerned with the Hilbert space of the theory modulo the chiral algebra. In the Kac–Moody case relevant for the WZW theory this means roughly that we forget the moding in the Kac–Moody generators. Naïvely, one might expect that after doing this one is left with the classical algebra. The deformation, however, is a consequence of the central extension of the Kac–Moody algebra which is crucial in determining its representations.

It is quite plausible that all solutions to the polynomial equations are given by combinations of quantum groups. Work in this direction is in progress.

I thank the organizers of the meeting for the opportunity to present this material in such a stimulating environment.

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